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A Note on the Buckling of an Elastic Plate Under the Influence of Simple Shear Flow

The linear von Kármán equation describing the buckling of an elastic flat plate due to a distributed deformation-dependent load is discussed. It is shown that, in the case of buckling under the influence of an over-passing simple shear flow, a detailed hydrodynamic analysis of the disturbance flow due to the buckling is not necessary, and the eigensolutions can be determined exclusively from the stress field of the unperturbed simple shear flow. Corrections to previous results for a circular membrane patch are made. [DOI: 10.1115/1.3197255]

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1 Introduction

We consider the buckling of an elastic plate in the xy plane due to a tangential distributed body force, as illustrated in Fig. 1. To compute the deflection along the z axis upon inception of buckling, $z=f(x,y)$, we work under the auspices of linear elastic stability of thin plates and shells and adopt the linear von Kármán equation,

$$\nabla^4 f \equiv \nabla^2\nabla^2 f = \frac{\partial^4 f}{\partial x^4} + 2\frac{\partial^4 f}{\partial x^2\,\partial y^2} + \frac{\partial^4 f}{\partial y^4} = \frac{h}{E_B}\left(\sigma_{xx}\frac{\partial^2 f}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 f}{\partial x\,\partial y} + \sigma_{yy}\frac{\partial^2 f}{\partial y^2} - p_n(x,y)\right) \quad (1)$$

where ∇ is the gradient in the xy plane, h is the plate thickness, E_B is a bending modulus, σ_{ij} is the in-plane stress tensor in the undeformed configuration, and p_n is the normal load in the deformed configuration (e.g., Refs. [1,2]).

If the normal load is due to a *deformation-independent* body force field in the xy plane, denoted by $\mathbf{b}(x,y)$, we use geometrical reasoning to find

$$p_n(x,y,f) = \mathbf{b}\cdot\nabla f = b_x\frac{\partial f}{\partial x} + b_y\frac{\partial f}{\partial y} \quad (2)$$

as illustrated in Fig. 1, yielding the Timoshenko equation:

$$\nabla^4 f = \frac{h}{E_B}\left(\sigma_{xx}\frac{\partial^2 f}{\partial x^2} + 2\sigma_{xy}\frac{\partial^2 f}{\partial x\,\partial y} + \sigma_{yy}\frac{\partial^2 f}{\partial y^2} - b_x\frac{\partial f}{\partial x} - b_y\frac{\partial f}{\partial y}\right) \quad (3)$$

Setting $\mathbf{b}=-\nabla\cdot\boldsymbol{\sigma}$, we obtain the compact form

$$\nabla^4 f = \frac{h}{E_B}\,\nabla\cdot(\boldsymbol{\sigma}\cdot\nabla f) \quad (4)$$

Adjustments must be made if a normal load is established due to the deformation. In linear stability analysis, the disturbance normal load can be approximated with the z component, yielding

$$p_n(x,y,f) = \mathbf{b}\cdot\nabla f + p_z^d \quad (5)$$

where the superscript d denotes the disturbance. As a physical application, we consider a distributed force field due to a simple shear flow along the x axis with velocity $u_x=\xi z$, where μ is the fluid viscosity and ξ is the shear rate. Luo and Pozrikidis [3–5] studied the membrane buckling by neglecting the disturbance traction due to the boundary deformation and using expression (2) with $b_x=\mu\xi$ and $b_y=0$, thus inadvertently discarding the perturbation normal load, p_z^d. As a consequence, their results apply only for a heavy vertical plate buckling due to its own weight.

In this note, we will show that, in the case of an elastic plate buckling under the influence of a simple shear flow, the correct expression for the normal load is

$$p_n = -\,\boldsymbol{\sigma}^{\mathrm{ssf}}:\mathbf{nn} \quad (6)$$

where $\boldsymbol{\sigma}^{\mathrm{ssf}}$ is the stress field of the simple shear flow given by

$$\boldsymbol{\sigma}^{\mathrm{ssf}} = \begin{bmatrix} 0 & 0 & \mu\xi \\ 0 & 0 & 0 \\ \mu\xi & 0 & 0 \end{bmatrix} \quad (7)$$

and $\mathbf{n}=[\partial f/\partial x, \partial f/\partial y, -1]$ is the linearized normal vector pointing downward, yielding

$$p_n = 2\mu\xi\frac{\partial f}{\partial x} \quad (8)$$

This expression differs from that arising from the superficial choice $b_x=\mu\xi$ and $b_y=0$ by a factor of 2. A redeeming consequence of this expression is that the hydrodynamic field does not need to be resolved in the deformed configuration for buckling thresholds to be established.

2 Normal Load in Simple Shear Flow

To demonstrate Eq. (6), we consider two-dimensional simple shear flow in the upper-half xy plane past a wavy wall with small-amplitude sinusoidal corrugations. The wall profile is described by the real or imaginary part of the function

$$y = f(x) = a\,\exp(ikx) \quad (9)$$

where a is the wall amplitude, $k=2\pi/L$ is the wave number, L is the wavelength, and i is the imaginary unit. The flow will be described in terms of a biharmonic stream function, ψ, and the harmonic pressure, p. Both can be expanded in perturbation series with respect to the dimensionless amplitude $\epsilon\equiv ka$,

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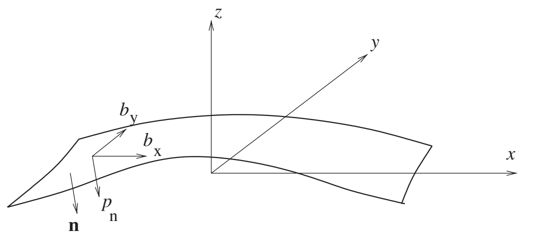

Fig. 1 Illustration of plate buckling under the influence of a body force; b is the body force field in the *xy* plane and p_n is the normal load

$$\psi(x,y) = \frac{1}{2}\xi y^2 + \epsilon\psi^{(1)}(x,y) + \cdots, \quad p(x,y) = \epsilon p^{(1)}(x,y) + \cdots \tag{10}$$

where ξ is the shear rate. The corresponding velocity components are

$$u_x = \frac{\partial\psi}{\partial y} = \xi y + \epsilon\frac{\partial\psi^{(1)}}{\partial y} + \cdots, \quad u_y = -\frac{\partial\psi}{\partial x} = -\epsilon\frac{\partial\psi^{(1)}}{\partial x} + \cdots \tag{11}$$

Next, we set

$$\psi^{(1)}(x,y) = \phi(y)\exp(ikx), \quad p^{(1)}(x,y) = \mu q(y)\exp(ikx) \tag{12}$$

where μ is the fluid viscosity, and the functions ϕ and q will be computed as part of the solution. Substituting these expressions in the x component of the Stokes equation, we find

$$p^{(1)} = -i\frac{\mu}{k}\left(\frac{\partial^3\psi^{(1)}}{\partial x^2\,\partial y} + \frac{\partial^3\psi^{(1)}}{\partial y^3}\right), \quad q(y) = ik^2\left(\frac{d\phi}{d\hat{y}} - \frac{d^3\phi}{d\hat{y}^3}\right) \tag{13}$$

where $\hat{y}\equiv ky$. To ensure that the perturbation flow due to the corrugations decays far from the wall, we set $\phi(\hat{y})=(A\hat{y}+B)\exp(-\hat{y})$. The no-slip and no-penetration boundary conditions require $\psi=0$ and $\partial\psi/\partial y=0$ over the wall, yielding $A=-\xi/k^2$ and $B=0$. Thus,

$$\phi = -\frac{\xi}{k}y\,\exp(-\hat{y}), \quad q = 2i\xi\,\exp(-\hat{y}) \tag{14}$$

Making substitutions, we derive the linear expressions

$$u_x = \xi y - \xi a(1-\hat{y})\exp(-\hat{y})\exp(ikx), \quad u_y = i\xi a\hat{y}\,\exp(-\hat{y})\exp(ikx)$$

$$p = 2i\mu\xi ka\,\exp(-\hat{y})\exp(ikx) \tag{15}$$

The linearized stress tensor is

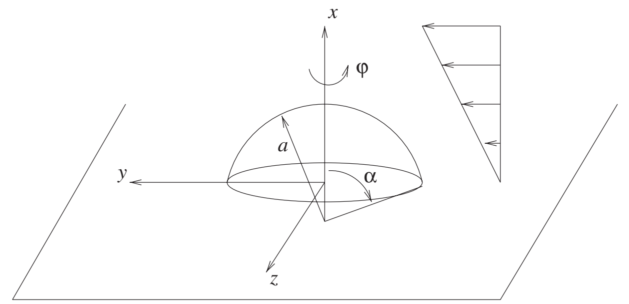

Fig. 2 Illustration of shear flow past a spherical protrusion attached to a plane wall

$$\boldsymbol{\sigma} = \mu\xi\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + 2\mu\xi ka\begin{bmatrix}i(\hat{y}-2) & 1-\hat{y}\\ 1-\hat{y} & -i\hat{y}\end{bmatrix}\exp(-\hat{y})\exp(ikx) \tag{16}$$

The linearized normal vector pointing downward is

$$\mathbf{n} = \begin{bmatrix}df/dx\\ -1\end{bmatrix} \tag{17}$$

yielding the linearized wall traction

$$\mathbf{f}_w \equiv -\boldsymbol{\sigma}(y=f)\cdot\mathbf{n} = \mu\xi\begin{bmatrix}1\\ -df/dx\end{bmatrix} + 2\mu\xi ka\begin{bmatrix}1\\ 0\end{bmatrix}\exp(ikx) \tag{18}$$

The linearized normal load emerging exclusively from the first term on right-hand side of Eq. (18) is $p_n=-\mathbf{f}_w\cdot\mathbf{n}=2\mu\xi df/dx$. The linearized tangential load is $p_t=\mu\xi[1+2ka\,\exp(ikx)]$. Using Fourier series expansions, we find that the result for the normal traction applies for a general wall profile described by an arbitrary function, $y=f(x)$. If the shear flow is directed parallel to the sinusoidal corrugations, the flow is unidirectional and the normal stress is constant, independent of the boundary geometry. Because shear flow past a small-amplitude three-dimensional topography can be linearly decomposed into a transverse and a longitudinal component over sinusoidal profiles by way of Fourier expansions, the predictions for two-dimensional flow remain valid.

To confirm these results, we consider simple shear flow past a shallow protrusion on an infinite plane wall, as shown in Fig. 2. The protrusion has the shape of a spherical sector of radius a and semi-angle α. Inertial effects are neglected and the motion of the fluid over the protrusion is governed by the equations of Stokes flow. The flow field was computed with high accuracy using a boundary-element method based on Fourier decomposition [6]. The solid lines in Figs. 3(*a*)–3(*c*) show the distribution of the

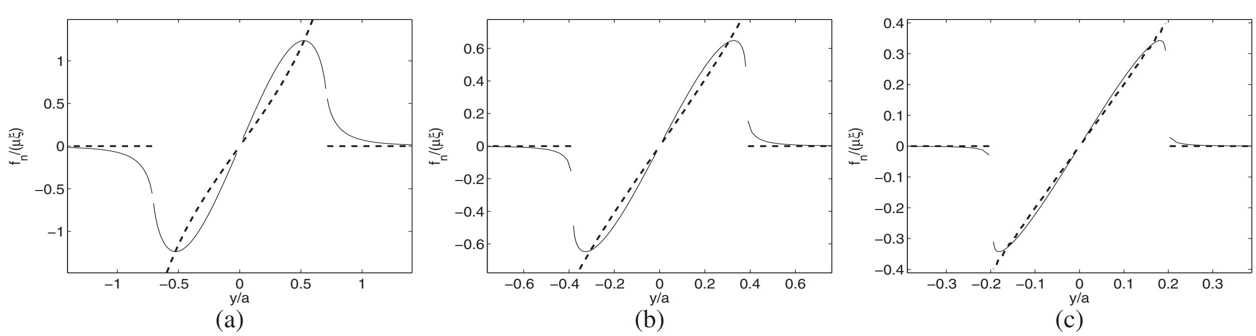

Fig. 3 Distribution of the normal component of the boundary traction along the trace of the wall and protrusion in the *xy* plane for semi-angle (*a*) $\alpha=\pi/4$, (*b*) $\pi/8$, and (*c*) $\pi/16$. The predictions of the linear theory are represented by the dashed lines.

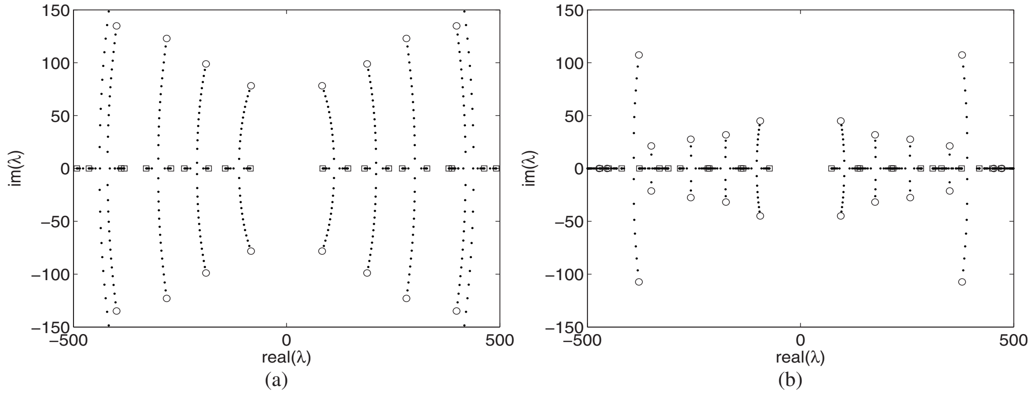

Fig. 4 Locus of the eigenvalues in the complex plane parameterized by the numerical coefficient β for Poisson ratio (*a*) ν=0.0 and (*b*) 0.5

normal component of the traction along the trace of the boundary in the xy plane, f_n, for aperture semi-angle $\alpha=\pi/4$, $\pi/8$, and $\pi/16$. The dotted lines represent the predictions of the linear theory discussed in this section, $f_n=-p_n=-2\mu\xi df/dx$. As the aperture angle α becomes smaller, the numerical results clearly converge to the asymptotic predictions.

3 Buckling of a Circular Plate

In the case of a clamped circular plate of radius a and thickness h deforming under the action of a uniform tangential load, τ, the displacement field in the xy plane is

$$v_x = \frac{\tau}{Eh}\frac{1-\nu^2}{3-\nu}(a^2-x^2-y^2), \quad v_y = 0 \tag{19}$$

and the associated stress field is

$$\sigma_{xx} = -\frac{2}{3-\nu}\frac{\tau}{h}x, \quad \sigma_{xy} = -\frac{1-\nu}{3-\nu}\frac{\tau}{h}y, \quad \sigma_{yy} = \nu\sigma_{xx} \tag{20}$$

where ν is the Poisson ratio. These expressions confirm that the streamwise component of the in-plane normal stress, σ_{xx}, is positive (tensile) on the upstream half, and negative (compressive) on the downstream half of the plate. The transverse component of the normal stress, σ_{yy}, is also positive or negative depending on the sign of the Poisson ratio. The von Kármán equation then becomes

$$\nabla^4 f = -\frac{\alpha}{a^3}\left[x\frac{\partial^2 f}{\partial x^2} + (1-\nu)y\frac{\partial^2 f}{\partial x\,\partial y} + \nu x\frac{\partial^2 f}{\partial y^2} + \frac{3-\nu}{2}2\frac{\partial f}{\partial x}\right] \tag{21}$$

where

$$\alpha \equiv \frac{2\tau a^3}{E_B(3-\nu)} \tag{22}$$

is a dimensionless parameter.

In the analysis of Luo and Pozrikidis [3–5], the coefficient of two in front of the last fraction on the right-hand side of Eq. (21) was not included. To illustrate the significance of this approximation, we consider the eigenvalues of a generalized equation where factor 2 in the last term on the right-hand side of Eq. (21) is replaced by a coefficient β. Applying the finite-element or spectral expansion method developed by Luo and Pozrikidis [3–5], we derive a generalized eigenvalue problem whose eigenvalues represent the inverse of the dimensionless critical buckling load, $\alpha=\lambda$. Figure 4 shows the locus of the eigenvalues in the complex plane for Poisson ratio ν=0 and 0.5, in the range of $1\le\beta\le 2$, computed with the finite-element method. When β=1, all eigenvalues shown with square symbols are real. As β increases toward 2, successive eigenvalues sequentially move off the real axis and become complex at critical thresholds. The eigenvalues for β=2 are shown as circles superposed on other symbols. We conclude that buckling thresholds either do not exist or are exceedingly high for a circular plate in simple shear flow.

4 Conclusion

We have presented a proper analysis for plate buckling due to an over-passing shear flow and made corrections to earlier work. We have shown that a detailed hydrodynamic analysis is not necessary for establishing the critical buckling thresholds, and the eigensolutions are determined solely from knowledge of the stress field of the unperturbed simple shear flow. These results apply only in the case of simple shear flow where the shear rate is uniform over the entire area of the plate. Under more general conditions, the hydrodynamic and plate bending equations are fully coupled and a more involved analysis is required.

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